Monopoles On S_F^2 From The Fuzzy Conifold

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Abstract

The intersection of the conifold $z_1^2 + z_2^2 + z_3^2 = 0$ and S^5 is a compact 3–dimensional manifold X^3. We review the description of X^3 as a principal $U(1)$ bundle over S^2 and construct the associated monopole line bundles. These monopoles can have only even integers as their charge. We also show the Kaluza–Klein reduction of X^3 to S^2 provides an easy construction of these monopoles. Using the analogue of the Jordon-Schwinger map, our techniques are readily adapted to give the fuzzy version of the fibration $X^3 \to S^2$ and the associated line bundles. This is an alternative new realization of the fuzzy sphere S_F^2 and monopoles on it.

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1 Introduction

A conifold is a complex manifold with a conical singularity at isolated point(s). In the neighbourhood of such a singular point the conifold is described by a quadratic in $\mathbb{C}^n$ [1]

$$\sum_{\alpha=1}^{n} z_\alpha^2 = 0, \tag{1.1}$$

where the singular point is chosen to be at the origin of $\mathbb{C}^n$. The conifold is a $(2n-2)$ dimensional space Y^{2n-2} which is smooth everywhere except at $z_\alpha = 0$.

Here we will specialise to the $n = 3$ case and show that the base of Y^4 is a fibre bundle over S^2. Interestingly our construction allows a natural fuzzification. The fuzzy version is a Jordon–Schwinger – like map and is a new construction of the fuzzy two–sphere S_F^2.

Let us briefly recall the properties of Y^{2n-2}. It is a noncompact Calabi-Yau manifold, with z_α transforming as vectors of $SO(n)$. It also admits an additional U(1) symmetry $z_\alpha \to e^{i\lambda} z_\alpha$, so the symmetry group is $SO(n) \times U(1)$. The base of Y^{2n-2} is a $(2n-3)$ dimensional manifold X^{2n-3} which is the intersection of Y^{2n-2} with the $S^{2n-1} \equiv \{\bar{z}_\alpha z_\alpha = \text{fixed}\}$. Y^{2n-2} is a cone over X^{2n-3}. The base X^{2n-3} is a compact Einstein manifold: $(R_{ij})_{X^{2n-3}} = (2n-4)\,(g_{ij})_{X^{2n-3}}$ while Y^{2n-2} is Ricci flat [2].

The $n = 4$ case has been studied in detail [1]. The symmetry group in this case is $SO(4) \times U(1) \simeq SU(2) \times SU(2) \times U(1)$. It can be shown that X^5 (also sometimes called $T^{1,1}$) is a $U(1)$ fibre bundle over $S^2 \times S^2$. What makes this manifold interesting is its deep connection to gauge-gravity duality (for example see [2, 3]). The fuzzy version of the $n = 4$ will be dealt seperately in another article [4].

Our interest in this article is in the $n = 3$ case. The symmetry group now is $SO(3) \times U(1)$ and the base X^3 is a 3–dimensional manifold [5]. We will show that there exist a "Hopf-like" map using the spin–1 representation of $SO(3)$. This map from X^3 to S^2 explicitly brings out the fact that X^3 is a $U(1)$ bundle over S^2. This $U(1)$ bundle is nontrivial and may be interpreted as a magnetic monopole at the centre of S^2. As we shall see, this monopole always has even integer charge. As one of our prime objective is to adapt the above construction to the fuzzy version, we will mainly use group theoretic techniques, refraining from using any differential geometry.

This article is organised as follows. In section 2 we review Y^4 and its base X^3, recalling their geometric properties and various symmetries. In section 3 we show that X^3 is a principal $U(1)$ bundle over S^2. The associated line bundles carry nontrivial monopole charge and describe topologically nontrivial configurations of a complex scalar field. In section 4 we show how these monopoles may be realized as arising from the Kaluza-Klein reduction of X^3 to S^2. This construction is very much in the spirit of [6, 7]. The fuzzy version is described in section 5. In section 5.1 we define the fuzzy conifold Y_F^4 and its base X_F^3 as certain subspaces of the Hilbert space of the 3–dimensional isotropic oscillator. In section 5.2 we show that the spin–1 matrices can be used to define a map $X_F^3 \to S_F^2$. To describe the fuzzy monopoles we will adopt the stategy of [8], directly constructing the sections of the fuzzy line bundles. Roughly speaking, these are "rectangular matrices" that map fuzzy sphere of one size to another.

2 The Conifold Y^4 And Its Base X^3

The conifold Y^4 [5] is a 4–dimensional manifold embedded in $\mathbb{C}^3$ with 3–complex coordinates z_α ($\alpha = 1, 2, 3$) satisfying

$$\mathcal{O}(z_\alpha) \equiv \sum_{\alpha=1}^{3} z_\alpha^2 = 0, \quad z_\alpha \in \mathbb{C}^3. \tag{2.1}$$

It is an $O(3)$ symmetric smooth manifold with conical singularity at a single point $z_\alpha = 0$, where the function $\mathcal{O}(z_\alpha)$ and its derivatives vanish:

$$\mathcal{O}(z_\alpha)|_{z_\alpha=0} = 0, \quad \left(\frac{\partial \mathcal{O}}{\partial z_\alpha}\right)_{z_\alpha=0} = 0. \tag{2.2}$$

The complex manifold Y^4 is the set of all lines passing through origin of $\mathbb{C}^3$ and hence a cone with a double singular point $z_\alpha = 0$ as its apex.

There is a scaling symmetry on Y^4: for any $\psi \in \mathbb{C}$ and any z_α obeying (2.1), ψz_α also solves (2.1). As we will see shortly, under this transformation $z_\alpha \to \psi z_\alpha$, the metric gets rescaled: $d\tilde{S}^2_{Y^4} \to |\psi|^2 d\tilde{S}^2_{Y^4}$. The space has a $SO(3) \times U(1)$ symmetry with an isolated Calabi-Yau singularity and the coordinates z_α transforms as vectors of $SO(3)$.

The intersection of Y^4 with the unit sphere $S^5 \subset \mathbb{C}^3$ is called the base X^3. It is a smooth 3–dimensional manifold devoid of any singularities and described by

$$\mathcal{O} \equiv z_1^2 + z_2^2 + z_3^2 = 0, \quad \bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{z}_3 z_3 = 1. \tag{2.3}$$

X^3 has $SO(3) \times U(1)$ symmetry and Y^4 is a cone over X^3. The manifold X^3 can be parametrized as [5]

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{2}} e^{2i\phi} \left(\cos^2 \frac{\theta}{2} - e^{2i\xi} \sin^2 \frac{\theta}{2}\right), \\ z_2 &= \frac{i}{\sqrt{2}} e^{2i\phi} \left(\cos^2 \frac{\theta}{2} + e^{2i\xi} \sin^2 \frac{\theta}{2}\right), \\ z_3 &= -\frac{1}{\sqrt{2}} e^{i(2\phi+\xi)} \sin\theta \end{aligned} \tag{2.4}$$

with $0 \leq \theta \leq \pi$, $-\pi \leq \xi \leq \pi$ and $0 \leq \phi \leq 2\pi$.

To parametrize Y^4, we need to add a radial coordinate r to the above parametrization. For $r \to \infty$ the metric on Y^4 can be written as in [3]

$$ds^2_{Y^4} = dr^2 + r^2 \left(a\left(d\phi + (1-\cos\theta)d\xi\right)^2 + b\left(d\theta^2 + \sin^2\theta d\xi^2\right)\right) \tag{2.5}$$

where a and b are constants. Demanding Ricci–flatness, we find that $a = 1/4 = b$, giving us

$$ds^2_{Y^4} = dr^2 + \frac{r^2}{4}\left(\left(d\phi + (1-\cos\theta)d\xi\right)^2 + \left(d\theta^2 + \sin^2\theta d\xi^2\right)\right). \tag{2.6}$$

The scaling $z_\alpha \to \psi z_\alpha$ can be exploited to rescale the $r \to \tilde{r} = \frac{r}{2}$. Then

$$ds^2_{Y^4} = 4d\tilde{r}^2 + \tilde{r}^2\left(\left(d\phi + (1-\cos\theta)d\xi\right)^2 + \left(d\theta^2 + \sin^2\theta d\xi^2\right)\right). \tag{2.7}$$

The angular part of this metric can be identified as the metric on the base X^3:

$$ds^2_{X^3} \equiv ds^2 = \left(d\theta^2 + \sin^2\theta d\xi^2\right) + \left((1-\cos\theta)d\xi + d\phi\right)^2. \tag{2.8}$$

3 X^3 Is a $U(1)$ Bundle Over S^2

Let us define a map $\Pi : \mathbb{C}^3 \to \mathbb{R}^3$

$$y_i = z^\dagger I_i z, \quad \bar{y}_i = y_i, \qquad z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{3.1}$$

where $i = 1, 2, 3$ and I_i are 3×3 matrices

$$(I_i)_{\alpha\beta} = -i\epsilon_{i\alpha\beta}, \qquad \text{where} \quad \alpha, \beta = 1, 2, 3 \tag{3.2}$$

These are the generators of the $SO(3)$ algebra in the fundamental representation:

$$I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \text{and} \quad I_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.3}$$

satisfying

$$[I_i, I_j] = i\epsilon_{ijk} I_k, \tag{3.4}$$

with the Casimir

$$\sum_{i=1}^{3} I_i I_i = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{3.5}$$

These y_i satisfy

$$\sum_{i=1}^{3} y_i y_i = \left(\sum_{\alpha=1}^{3} \bar{z}_\alpha z_\alpha \right)^2 - \sum_{\alpha,\beta=1}^{3} \bar{z}_\alpha^2 z_\beta^2 = \left(\sum_{\alpha=1}^{3} \bar{z}_\alpha z_\alpha \right)^2 - \bar{\mathcal{O}}\mathcal{O}. \tag{3.6}$$

So if $z_\alpha \in X^3$, then

$$\sum_{i=1}^{3} y_i y_i = 1. \tag{3.7}$$

This space is the unit sphere S^2 and thus (3.1) is a map $\Pi : X^3 \to S^2$. Using the (2.4), we can explicitly write Π as

$$\begin{aligned} y_1 = &\quad -i\left(\bar{z}_2 z_3 - \bar{z}_3 z_2\right) = &\quad \sin\theta\cos\xi, \\ y_2 = &\quad -i\left(\bar{z}_3 z_1 - \bar{z}_1 z_3\right) = &\quad \sin\theta\sin\xi, \\ y_3 = &\quad -i\left(\bar{z}_1 z_2 - \bar{z}_2 z_1\right) = &\quad \cos\theta, \end{aligned} \tag{3.8}$$

where $0 \leq \theta \leq \pi$ and $-\pi \leq \xi \leq \pi$.

X^3 has a $U(1)$ symmetry as (2.3) is invariant under $z_\alpha \to e^{i\lambda} z_\alpha$. For $z_\alpha \in X^3$, Π is also invariant under this transformation. It is evident from (3.8) that the sphere S^2 is independent of ϕ. This means Π maps circles S^1 on X^3 to points on S^2. X^3 is thus a $U(1)$ bundle over S^2:

$$\begin{array}{ccc} U(1) & \rightarrow & X^3 \\ & & \downarrow \\ & & S^2. \end{array} \tag{3.9}$$

The angles θ and ξ are the coordinates of S^2 and the fibre cordinate is ϕ.

It is useful to explicitly describe the coordinate charts that we will use on S^2. To this end, we define the complex functions

$$\begin{aligned} w_1^N &= \frac{1}{\sqrt{2}}\left(\cos^2\frac{\theta}{2} - e^{2i\xi}\sin^2\frac{\theta}{2}\right), \\ w_2^N &= \frac{i}{\sqrt{2}}\left(\cos^2\frac{\theta}{2} + e^{2i\xi}\sin^2\frac{\theta}{2}\right), \\ w_3^N &= -\frac{1}{\sqrt{2}}e^{i\xi}\sin\theta. \end{aligned} \tag{3.10}$$

These functions are well–defined at all points on S^2 except for $\theta = \pi$ (the South Pole S). It is easy to see that (3.10) is obtained by setting $\phi = 0$ in (2.4). Let us denote this coordinate chart as U_N. To describe the coordinate chart U_S that includes the South Pole we set $\phi = -\xi$ in (2.4) to obtain

$$\begin{aligned} w_1^S &= \frac{1}{\sqrt{2}}\left(e^{-2i\xi}\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right), \\ w_2^S &= \frac{i}{\sqrt{2}}\left(e^{-2i\xi}\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right), \\ w_3^S &= -\frac{1}{\sqrt{2}}e^{-i\xi}\sin\theta. \end{aligned} \tag{3.11}$$

These functions are well–defined at all points on S^2 except at $\theta = 0$ (the North Pole N). On the overlapping region $U_N \cap U_S$,

$$w_\alpha^N = e^{-2i\xi} w_\alpha^S. \tag{3.12}$$

It is important to note that the (3.10) and (3.11) are not the standard stereographic projection maps.

Now let us describe the topologically non–trivial configurations of a complex scalar field on this S^2. A complex scalar field on $S_N^2 \equiv S^2 - \{S\}$ is a function of w_α^N

$$\Phi_N = \sum c_{n_1 n_2 n_3 n_1' n_2' n_3'} (\bar{w}_1^N)^{n_1'} (\bar{w}_2^N)^{n_2'} (\bar{w}_3^N)^{n_3'} (w_1^N)^{n_1} (w_2^N)^{n_2} (w_3^N)^{n_3} \tag{3.13}$$

while on the other patch $S_S^2 \equiv S^2 - \{N\}$, it is a function of w_α^S

$$\Phi_S = \sum c_{n_1 n_2 n_3 n_1' n_2' n_3'} (\bar{w}_1^S)^{n_1'} (\bar{w}_2^S)^{n_2'} (\bar{w}_3^S)^{n_3'} (w_1^S)^{n_1} (w_2^S)^{n_2} (w_3^S)^{n_3}. \tag{3.14}$$

If $k = n_1' + n_2' + n_3' - n_1 - n_2 - n_3 =$ fixed, then in the region $S_N^2 \cap S_S^2$, Φ_N is related to Φ_S as

$$\Phi_N = e^{-2i(n_1' + n_2' + n_3' - n_1 - n_2 - n_3)\xi}\Phi_S = e^{-i\kappa\xi}\Phi_S, \quad \kappa = 2k. \tag{3.15}$$

We recognise the phase in the above equation as the gauge transformation relating Φ_N and Φ_S. This gauge transformation arises from a gauge field A_μ with

$$\begin{aligned} A_\mu^N &= -i\tfrac{\kappa}{2}\bar{w}_\alpha^N\left(\partial_\mu w_\alpha^N\right) \quad \text{on } S_N^2, \\ A_\mu^S &= -i\tfrac{\kappa}{2}\bar{w}_\alpha^S\left(\partial_\mu w_\alpha^S\right) \quad \text{on } S_S^2, \\ \text{and} \quad A_\mu^N &= A_\mu^S + ie^{i\kappa\xi}\left(\partial_\mu e^{-i\kappa\xi}\right) \quad \text{on } S_N^2 \cap S_S^2 \end{aligned} \tag{3.16}$$

where $\mu = \theta, \xi$. Explicit computation gives

$$A_\theta^N = 0, \quad A_\xi^N = \frac{\kappa}{2}(1-\cos\theta); \quad A_\theta^S = 0, \quad A_\xi^S = -\frac{\kappa}{2}(1+\cos\theta). \tag{3.17}$$

The connection one-forms are

$$A^N = A_\theta^N d\theta + A_\xi^N d\xi = \frac{\kappa}{2}(1-\cos\theta)d\xi; \quad A^S = A_\theta^S d\theta + A_\xi^S d\xi = -\frac{\kappa}{2}(1+\cos\theta)d\xi \tag{3.18}$$

In the overlapping region $S_N^2 \cap S_S^2$, A_N and A_S are related as $A^N - A^S = \kappa d\xi$ where κ is even integer.

We denote by $\mathcal{H}_\kappa$ the space of these complex scalar fields with a fixed κ. The elements Φ of $\mathcal{H}_\kappa$ are eigenfunctions of the operator K_0 with eigenvalue $\frac{\kappa}{2}$:

$$K_0\Phi \equiv \sum_{\alpha=1}^{3}\left(\bar{w}_\alpha \frac{\partial}{\partial \bar{w}_\alpha} - w_\alpha \frac{\partial}{\partial w_\alpha}\right)\Phi = \frac{\kappa}{2}\Phi. \tag{3.19}$$

We will therefore call this operator as the topolological charge operator. $\mathcal{H}_\kappa$ is the linear space of sections Φ with has a topological charge κ.

The differential operators $J_i = -i\epsilon_{ijk}y_j\frac{\partial}{\partial y_k}$ can be written in terms of w_α:

$$\begin{aligned}
J_1 &= -i\left(\bar{w}_2\frac{\partial}{\partial \bar{w}_3} - w_3\frac{\partial}{\partial w_2} - \bar{w}_3\frac{\partial}{\partial \bar{w}_2} + w_2\frac{\partial}{\partial w_3}\right)\\
J_2 &= -i\left(\bar{w}_3\frac{\partial}{\partial \bar{w}_1} - w_1\frac{\partial}{\partial w_3} - \bar{w}_1\frac{\partial}{\partial \bar{w}_3} + w_3\frac{\partial}{\partial w_1}\right)\\
J_3 &= -i\left(\bar{w}_1\frac{\partial}{\partial \bar{w}_2} - w_2\frac{\partial}{\partial w_1} - \bar{w}_2\frac{\partial}{\partial \bar{w}_1} + w_1\frac{\partial}{\partial w_2}\right).
\end{aligned} \tag{3.20}$$

The J_i's act on $\mathcal{H}_\kappa$ and map $\mathcal{H}_\kappa \to \mathcal{H}_\kappa$. In $\mathcal{H}_\kappa$, J_i's satisfy the $SU(2)$ algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{3.21}$$

The topological charge operator K_0 commutes with J_i:

$$[K_0, J_i] = 0. \tag{3.22}$$

In $\mathcal{H}_\kappa$ we can choose the eigenfunctions of J_3 and J_ij_i as a basis to expand any $\Phi \in \mathcal{H}_\kappa$.

Now let us construct these basis functions. The function $h = (\bar{w}_1 + i\bar{w}_2)^l(w_1 + iw_2)^n$ is an element of $\mathcal{H}_\kappa$ with $\kappa = 2(l-n)$ as

$$K_0 h = (l-n)h. \tag{3.23}$$

This function h satisfies

$$J_+h = 0, \quad J_3h = (l+n)h \tag{3.24}$$

and so h is the highest weight vector of the (3.21) with $j = (l+n)$. We denote this highest weight vector h by $\Phi^j_{\kappa,m=j}$, from which the lower weight vectors can be obtained by repeated application of the J_-:

$$(J_-)^{(j-m)}\Phi^j_{\kappa,m=j} = N^j_{\kappa,m}\Phi^j_{\kappa,m}, \quad -j \le m \le j. \tag{3.25}$$

The constants $N^j_{\kappa,m}$ can be evaluated explicitly but are unnecessary for our purposes, and we will not do so here.

The value of j is given by (3.24)

$$j \equiv l + n = \frac{\kappa}{2} + 2n. \tag{3.26}$$

As l and n take values $0, 1, 2, 3 \ldots$ and as κ can have only even integer value, while j takes all integer values greater that $\frac{\kappa}{2}$:

$$j = \frac{\kappa}{2}, \frac{\kappa}{2} + 2, \frac{\kappa}{2} + 4, \ldots \tag{3.27}$$

The set $\{\Phi^j_{\kappa,m}\}$ spans $\mathcal{H}_\kappa$ and any element Φ of $\mathcal{H}_\kappa$ can be expressed as

$$\Phi = \sum_{j=\frac{\kappa}{2}}^{\infty} \sum_{m=-j}^{j} c^j_{\kappa,m} \Phi^j_{\kappa,m}, \qquad c^j_{\kappa,m} \in \mathbb{C}. \tag{3.28}$$

These elements of $\mathcal{H}_\kappa$ are identified as sections of the line bundle with topological charge κ (= even integer).

4 Monopoles From Kaluza-Klein Reduction $X^3 \to S^2$

Interestingly, the principal fibre bundle of the previous discussion can be obtained by the Kaluza-Klein reduction of the metric on X^3. The metric (2.8) can be written as $ds^2 = \tilde{g}_{ab} d\eta^a d\eta^b$ where

$$\tilde{g}_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta + 4\sin^2\frac{\theta}{2} & 2\sin^2\frac{\theta}{2} \\ 0 & 2\sin^2\frac{\theta}{2} & 1 \end{pmatrix}; \quad \eta^1 = \theta, \eta^2 = \xi, \eta^3 = \phi. \tag{4.1}$$

It is an Einstein metric since

$$\tilde{R}_{ab} = \frac{1}{2}\tilde{g}_{ab} \tag{4.2}$$

and it solves Einstein equations with a positive cosmological constant.

The metric (4.1) already has the convenient Kaluza-Klein form

$$\tilde{g}_{ab} = \begin{pmatrix} g_{\mu\nu} + gA_\mu A_\nu & gA_\mu \\ gA_\nu & g \end{pmatrix}, \quad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \tag{4.3}$$

with with one extra compact dimension ϕ and the dilaton g set to 1. Inspecting (4.1) immediately tells us that $g_{\mu\nu}$ is the metric on S^2_N and the gauge fields on this local patch are $A^N_\theta = 0$, $A^N_\xi = 2\sin^2\frac{\theta}{2} = (1-\cos\theta)$. The field strength $F^N = dA^N$ is

$$F^N_{\theta\xi} = \frac{\partial A^N_\xi}{\partial\theta} - \frac{\partial A^N_\theta}{\partial\xi} = \sin\theta. \tag{4.4}$$

This corresponds to a constant radial magnetic field $B^N_r = *F^N$ and hence a monopole of charge $\kappa = 2$ at the center of the sphere.

Similar computations on S^2_S give us $A^S_\theta = 0$, $A^S_\xi = -(1-\cos\theta)$. A^N_ξ and A^S_ξ are related by the gauge transformation $A^N_\xi = A^S_\xi - g\partial_\xi g^{-1}$, where $g = e^{2i\xi} \in U(1)$. This is a simplified version of the Kaluza-Klein monopole (for the standard KK monopole see [6, 7]) .

Notice that the metric on X^3 is in the KK-form. Further both X^3 and S^2 are Einstein manifolds. This observation provides us with a handle for generalizing this construction to other manifolds and dimensions. Consider the Kaluza-Klein reduction of $\mathcal{M}^{d+1}$ to $\mathcal{M}^d$ where both $\mathcal{M}^{d+1}$ and $\mathcal{M}^d$ are compact Einstein manifolds. As we shall see below, the "Einstein condition" (the Ricci tensor is proportional to the metric) leads to stringent conditions on the gauge field. When $\mathcal{M}^d$ is S^2 it also leads to a relation between the monopole charge, size of S^2, the dilaton g and the cosmological constant.

Consider a $(d+1)$–dimensional manifold $\mathcal{M}^{d+1}$ with metric $\tilde{g}_{ab}$ in the form (4.3). The $g_{\mu\nu}$ is identified as the metric on a d–dimensional manifold $\mathcal{M}^d$. There is one extra-dimension which we assume is compact, and that $\tilde{g}_{ab}$ is independent of the coordinate of the extra dimension (Kaluza's cylinder condition). Let $\tilde{g}_{d+1,d+1} \equiv g$ be a constant. The components of the Ricci tensor are

$$\begin{aligned} \tilde{R}_{d+1,d+1} &= \frac{g^2}{4} F^{\sigma\beta}F_{\sigma\beta}, \quad \tilde{R}_{\mu,d+1} = \frac{g^2}{4}F^{\sigma\beta}F_{\sigma\beta}A_\mu = \tilde{R}_{d+1,\mu}, \\ \tilde{R}_{\mu\nu} &= R_{\mu\nu} + \frac{g^2}{4}F^{\sigma\beta}F_{\sigma\beta}A_iA_j - \frac{g}{4}F^{\sigma\beta}\left(g_{\mu\beta}F_{\sigma\nu} + g_{\nu\beta}F_{\sigma\mu}\right) \end{aligned} \tag{4.5}$$

where $F_{\sigma\beta} = \partial_\sigma A_\beta - \partial_\beta A_\sigma$ and $F^{\sigma\beta} = g^{\rho\sigma}g^{\alpha\beta}F_{\rho\alpha}$.

Since both $\mathcal{M}^{d+1}$ and $\mathcal{M}^d$ are Einstein manifolds (with different cosmological constants!)

$$\tilde{R}_{ab} = c\tilde{g}_{ab}, \quad R_{\mu\nu} = c_0 g_{\mu\nu}. \tag{4.6}$$

It is easy to see that the gauge fields must satisfy the conditions

$$F^{\sigma\beta}F_{\sigma\beta} = \frac{4c}{g}, \tag{4.7}$$

$$F^{\sigma\beta}\left(g_{\mu\beta}F_{\sigma\nu} + g_{\nu\beta}F_{\sigma\mu}\right) = \frac{4}{g}(c_0 - c)g_{\mu\nu}. \tag{4.8}$$

Multiplying (4.8) by $g^{\mu\nu}$ and summing over μ and ν, we get

$$F^{\sigma\beta}F_{\sigma\beta} = \frac{2}{g}(c_0 - c)d \tag{4.9}$$

yielding

$$c_0 = \frac{d+2}{d}c. \tag{4.10}$$

Moreover, as $\mathcal{M}^d$ is compact, (4.7) implies that the electromagnetic action

$$S_{EM} \propto \int d(Vol)F^{\sigma\beta}F_{\sigma\beta} = \frac{4c}{g}Volume \tag{4.11}$$

is finite.

Let us specialize to the case when $\mathcal{M}^{d+1}$ is 3–dimensional and $\mathcal{M}^d = S^2$. For example $\mathcal{M}^{d+1}$ could be S^3 (the standard Hopf fibration) or X^3 (our (3.8)). Then (4.10) gives $c_0 = 2c$ and the metric on S^2 is

$$g_{\mu\nu} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}, \quad R = \begin{cases} \frac{1}{2} \text{ for } S^3 \to S^2 \\ \\ 1 \text{ for } X^3 \to S^2. \end{cases} \tag{4.12}$$

From (4.8), we can write

$$F_{12} = \sqrt{\frac{2c}{g}\frac{g_{11}}{g^{22}}} = R^2\sqrt{\frac{2c}{g}}\sin\theta. \tag{4.13}$$

So, there is a radial magnetic field which corresponds to a magnetic monopole of charge $\kappa = 2R^2\sqrt{\frac{2c}{g}}$ at the centre of the sphere. For S^3 this gives $F_{12} = \frac{1}{2}\sin\theta$ and $\kappa = 1$ where as for X^3 we get $F_{12} = \sin\theta$ and $\kappa = 2$.

5 Fuzzy Fibre Bundle

Fuzzy spaces are quantized symplectic manifolds which are co-adjoint orbits of Lie groups. Often such spaces can be represented by a set of bosonic oscillators (see [9]). Such spaces are topics of interest to physicists and mathematicians especially in context of finite dimensional approximations of quantum field theories. Fuzzy spaces have encrypted topological information and it is interesting to study the classical solutions like instantons, solitons, monopoles etc. which have topological origin (for example see [10–16]).

The fuzzy n–dimensional complex plane $\mathbb{C}^n_F$ is represented by n independent bosonic oscillators

$$[\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, 3 \ldots n. \tag{5.1}$$

These operators act on the Hilbert space of the n bosonic oscillators. Restricting to appropiate subspaces of this Hilbert space gives us other fuzzy spaces like S^{2n-1}_F, $\mathbb{CP}^n_F$ and so on [9, 17, 18].

For the $n = 2$ the Fock space of states on which these operators operate is spanned by the states $|n_1, n_2\rangle$ – the eigenstates of the number operator $N = \hat{a}^\dagger_1\hat{a}_1 + \hat{a}^\dagger_2\hat{a}_2$. In the subspaces $f_n = \{|n_1, n_2\rangle, n_1 + n_2 = n = \text{fixed}\}$, the number operator N takes a constant value and the restriction to this subspace defines the fuzzy manifold S^3_F. The Jordon-Schwinger realization of the $SU(2)$ algebra is an operator map $L_i = \frac{1}{2}\hat{a}^\dagger_\alpha(\sigma_i)^{\alpha\beta}\hat{a}_\beta$ such that

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \quad \sum_{i=1}^{3} L_iL_i = \frac{N}{2}\left(\frac{N}{2} + 1\right). \tag{5.2}$$

In the subspace f_n, the Casimir L_iL_i is fixed and resultant fuzzy space is S^2_F. This map $S^3_F \to S^2_F$ is the noncommutative Hopf fibration.

Though standard differential geometric tools are unavailable for these discrete spaces, much topological information can still be extracted indirectly by studying the the complex line bundles. A particularly simple approach has been developed by [8], where the group action of $SU(2)$ is used to identify the fuzzy line bundles. It has gained some

special attention due to its simplicity and the lucid connection of the approach with the continuous case.

The idea of noncommutative Hopf map can be generalized to higher dimensional noncommutative spaces. In particular, we are interested in a"Hopf-like" construction which relates the fuzzy conifold Y_F^4 with S_F^2. We show below that the techniques of [8] can be adapted to describe the fuzzy fibration $X_F^3 \to S_F^2$, construct the corresponding line bundles and identify the monopole charges.

5.1 The Fuzzy Conifold Y_F^4 And The Base X_F^3

$\mathbb{C}_F^3$ is described by the algebra of three independent oscillators

$$[\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}, \qquad \text{where} \quad \alpha, \beta = 1, 2, 3. \tag{5.3}$$

which acts on a space $\mathcal{F}$ spanned by the eigenstates of the *number* operators $\hat{N}_\alpha \equiv \hat{a}_\alpha^\dagger \hat{a}_\alpha$:

$$\mathcal{F} = Span \left\{ |n_1, n_2, n_3\rangle : n_\alpha = 0, 1, 2.....\right\}. \tag{5.4}$$

The total number operator is

$$\hat{N} \equiv \sum_{\alpha=1}^{3} \hat{N}_\alpha.$$

In analogy with (2.1) let us define the operator $\hat{\mathcal{O}}$ as

$$\hat{\mathcal{O}} \equiv \hat{a}_1^2 + \hat{a}_2^2 + \hat{a}_3^2 \tag{5.5}$$

which has as its kernel

$$\ker(\hat{\mathcal{O}}) = \left\{ |n_1, n_2, n_3\rangle : \ \hat{\mathcal{O}}|n_1, n_2, n_3\rangle = 0 \right\}. \tag{5.6}$$

We define the fuzzy conifold Y_F^4 as the restriction of the action of the operators $\hat{a}_\alpha$ to $\ker(\hat{\mathcal{O}})$. It is easy to see why this definition is appropriate. Let $|z_1, z_2, z_3\rangle$ be the standard coherent states of 3–dimensional oscillator

$$|z_1, z_2, z_3\rangle = N(z_\alpha, \bar{z}_\alpha) e^{z_\alpha \hat{a}_\alpha^\dagger} |0, 0, 0\rangle. \tag{5.7}$$

Then

$$\hat{\mathcal{O}}|z_1, z_2, z_3\rangle = \left(z_1^2 + z_2^2 + z_3^2\right) |z_1, z_2, z_3\rangle \tag{5.8}$$

This implies that if $|z_1, z_2, z_3\rangle \in \ker(\hat{\mathcal{O}})$, the complex numbers z_α obey the conifold condition (2.1).

The operators

$$\hat{\chi}_\alpha = \hat{a}_\alpha \frac{1}{\sqrt{\hat{N}}}, \quad \hat{\chi}_\alpha^\dagger = \frac{1}{\sqrt{\hat{N}}} \hat{a}_\alpha^\dagger \tag{5.9}$$

satisfy

$$\hat{\chi}_\alpha^\dagger \hat{\chi}_\alpha = 1. \tag{5.10}$$

We exclude the state $|0, 0, 0\rangle$ from the domain of $\hat{\chi}_\alpha$ so that $\hat{\chi}_\alpha$ is well defined . Then (5.10) gives us the fuzzy 5–sphere S_F^5.

The operator

$$\hat{\mathcal{O}}' \equiv \sum_{\alpha=1}^{3} \hat{\chi}_\alpha^2 = \frac{1}{\sqrt{(\hat{N}+1)(\hat{N}+2)}} \hat{\mathcal{O}} \tag{5.11}$$

obviously vanishes on $\ker(\hat{\mathcal{O}})$. The restriction of $\hat{\chi}_\alpha$ to $\ker(\hat{\mathcal{O}})$ defines for us X_F^3, the fuzzy version of X^3. One may think of X_F^3 as the intersection of Y_F^4 with S_F^5.

The continuum X^3 can be recovered in the limit $\hat{N} \to \infty$ of X_F^3. It is simplest to see this using the coherent states (5.7) but we will skip the details here.

5.2 Fuzzy Two–Sphere S_F^2 And The Noncommutative Fibre Bundle

With the matrices (3.3), we can write the analogue of the map (3.1):

$$\hat{y}_i = \hat{\chi}^\dagger I_i \hat{\chi} = \frac{1}{\hat{N}} \hat{L}_i, \quad \text{where} \quad \hat{L}_i = \hat{a}_\alpha^\dagger (I_i)_{\alpha\beta} \hat{a}_\beta \quad \text{and} \quad \hat{\chi} = \begin{pmatrix} \hat{\chi}_1 \\ \hat{\chi}_2 \\ \hat{\chi}_3 \end{pmatrix}. \tag{5.12}$$

The $\hat{L}_i$'s satisfy

$$[\hat{L}_i, \hat{L}_j] = i\epsilon_{ijk} \hat{L}_k, \quad [\hat{L}_i, \hat{N}] = 0. \tag{5.13}$$

The Casimir can be conveniently expressed as

$$\hat{C} = \hat{L}_i \hat{L}_i = \hat{N}\left(\hat{N}+1\right) - \hat{\mathcal{O}}^\dagger \hat{\mathcal{O}}. \tag{5.14}$$

As $\hat{L}_i$ and $\hat{N}$ commute, it is easy to see that

$$\hat{y}_i \hat{y}_i = \frac{1}{\hat{N}^2} \hat{L}_i \hat{L}_i = \left(1 + \frac{1}{\hat{N}}\right) - \frac{1}{\hat{N}^2} \hat{\mathcal{O}}^\dagger \hat{\mathcal{O}}. \tag{5.15}$$

In $\ker(\hat{\mathcal{O}})$, $\hat{C}$ and $\hat{y}_i \hat{y}_i$ are simply

$$\hat{C}|_{\ker(\hat{\mathcal{O}})} = \hat{N}\left(\hat{N}+1\right), \qquad \hat{y}_i \hat{y}_i|_{\ker(\hat{\mathcal{O}})} = \left(1 + \frac{1}{\hat{N}}\right). \tag{5.16}$$

It is convenient to decompose $\mathcal{F}$ into subspaces $\mathcal{F}_n$ in which $\hat{N}$ takes a fixed value n:

$$\mathcal{F}_n = \{|n_1, n_2, n_3\rangle : n_1 + n_2 + n_3 = n\}, \quad \mathcal{F} = \oplus_n \mathcal{F}_n. \tag{5.17}$$

The dimension d_n of $\mathcal{F}_n$ is $\frac{(n+1)(n+2)}{2}$.

$\tilde{\mathcal{F}}_n$ is the subspace of $\mathcal{F}_n$ defined as

$$\tilde{\mathcal{F}}_n = \mathcal{F}_n \cap \ker(\hat{\mathcal{O}}). \tag{5.18}$$

It has the nice property that both $\hat{\mathcal{O}}$ vanishes and the value of $\hat{N}$ is fixed. So in $\tilde{\mathcal{F}}_n$

$$\hat{y}_i \hat{y}_i = \left(1 + \frac{1}{n}\right) \mathbb{1}. \tag{5.19}$$

When restricted to $\tilde{\mathcal{F}}_n$, the Casimir $\hat{C}$ takes the fixed value $n\,(n+1)$ and $\tilde{\mathcal{F}}_n$ is the carrier space for the $(2n+1)$ dimensional UIR of the $SU(2)$. As n takes integer values, only the odd dimensional representations occur in this construction. This construction was first done in [19, 20] in the context of ferromagnetism.

Thus the algebra generated by $\hat{y}_i$'s restricted to $\tilde{\mathcal{F}}_n$ is the fuzzy two–sphere S^2_F, and (5.12) is a map $X^3_F \to S^2_F$. The $n \to \infty$ is the commutative limit and in this limit as $\hat{y}_i\hat{y}_i \to 1$, we recover S^2.

We will now use the $SU(2)$ group theory to construct the noncommutative fibre bundles on this S^2_F. Our strategy will be similar to the one in [8].

Let $\mathcal{H}_{nl}$ be the space of linear operators Φ which map $\tilde{\mathcal{F}}_n$ to $\tilde{\mathcal{F}}_l$:

$$\Phi : \tilde{\mathcal{F}}_n \to \tilde{\mathcal{F}}_l, \quad \Phi \in \mathcal{H}_{nl}. \tag{5.20}$$

The operators Φ can be represented by rectangular matrices of size $(2l+1) \times (2n+1)$. The spaces $\mathcal{H}_{nn}$ are $(2n+1)^2$ dimensional noncommutative algebras $\mathcal{A}_n$ which map $\tilde{\mathcal{F}}_n \to \tilde{\mathcal{F}}_n$. The space $\mathcal{H}_{nl}$ is a noncommutative bimodule: it is left $\mathcal{A}_l$–module and a right $\mathcal{A}_n$–module.

Rotations are generated in $\mathcal{H}_{nn}$ by the adjoint action of $\hat{L}^{(n)}_i$:

$$Ad(\hat{L}_i)\Phi \equiv \hat{\mathcal{L}}_i\Phi \equiv [\hat{L}^{(n)}_i, \Phi], \quad \Phi \in \mathcal{H}_{nn} \tag{5.21}$$

and

$$[\hat{\mathcal{L}}_i, \hat{\mathcal{L}}_j] = i\epsilon_{ijk}\hat{\mathcal{L}}_k. \tag{5.22}$$

Since $\tilde{\mathcal{F}}_n$ is the carrier space for the $(2n+1)$–dimensional UIR, the $\hat{L}^{(n)}_i$ above are the usual $(2n+1) \times (2n+1)$ matrices.

On the bimodules $\mathcal{H}_{nl}$, the generators of the $SU(2)$ algebra acts as

$$\hat{\mathcal{L}}_i\Phi = \hat{L}^{(l)}_i\Phi - \Phi\hat{L}^{(n)}_i. \tag{5.23}$$

This action of $SU(2)$ corresponds to the direct product $l \otimes n$ of the two UIRs l and n. The elements of $\mathcal{H}_{nl}$ can therefore be expanded in terms of the eigenfunctions of $\hat{\mathcal{L}}_3$ and $\hat{\mathcal{L}}_i\hat{\mathcal{L}}_i$ belonging to the irreducible representations in the decomposition of $l \otimes n$:

$$l \otimes n = |l-n| \oplus |l-n|+1 \oplus \ldots \oplus (l+n). \tag{5.24}$$

We denote the minimum and the maximum values in this series as

$$|l-n| \equiv \frac{\kappa}{2}, \quad l+n \equiv J. \tag{5.25}$$

Below we construct these basis functions explictly. The sections of the fuzzy line bundle can be expanded in terms of these basis functions.

The operator

$$\hat{h} = N_{\tilde{l}\tilde{n}}(\hat{\chi}^\dagger_1 + i\hat{\chi}^\dagger_2)^{\tilde{l}}(\hat{\chi}_1 + i\hat{\chi}_2)^{\tilde{n}}, \quad N_{\tilde{l}\tilde{n}} = \text{constant} \tag{5.26}$$

is an element of $\mathcal{H}_{nl}$ if $0 \leq \tilde{n} \leq n$ and $\tilde{l} - \tilde{n} = l - n \equiv \frac{\kappa}{2}$. Let us define a new set of oscillators $\{\hat{A}_1, \hat{A}_2, \hat{A}_3\}$ as

$$\begin{pmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} \tag{5.27}$$

and $\hat{\xi}_\alpha \equiv \hat{A}_\alpha \frac{1}{\sqrt{\hat{N}}}$. It is easy to check that

$$[\hat{A}_\alpha, \hat{A}^\dagger_\beta] = \delta_{\alpha\beta}. \quad \hat{A}^\dagger_\alpha \hat{A}_\alpha = \hat{a}^\dagger_\alpha \hat{a}_\alpha = \hat{N}, \quad \xi^\dagger_\alpha \xi_\alpha = 1. \tag{5.28}$$

In terms of these new oscillators,

$$\hat{L}_+ = \sqrt{2}(\hat{A}^\dagger_3 \hat{A}_1 - \hat{A}^\dagger_2 \hat{A}_3), \quad \hat{L}_- = \hat{L}^\dagger_+, \qquad \text{and} \quad \hat{L}_3 = (\hat{A}^\dagger_1 \hat{A}_1 - \hat{A}^\dagger_2 \hat{A}_2). \tag{5.29}$$

$\hat{L}_\pm, \hat{L}_3$ are defined in the appendix (A.8, A.9, A.10). The operator $\hat{h} = N'_{\tilde{l}\tilde{n}} (\hat{\xi}^\dagger_2)^{\tilde{l}} (\hat{\xi}_1)^{\tilde{n}}$ satisfies

$$\hat{\mathcal{L}}_+ \hat{h} \equiv [\hat{L}_+, \hat{h}] = 0 \tag{5.30}$$

$$\hat{\mathcal{L}}_3 \hat{h} \equiv [\hat{L}_3, \hat{h}] = \left(\tilde{l} + \tilde{n}\right) \hat{h} \tag{5.31}$$

making $\hat{h}$ is the highest weight vector of the $SU(2)$ representation with $j = (\tilde{l} + \tilde{n})$. We denote this highest weight vector by $\Phi^j_{J,\kappa,j}$. The lower weight vectors can be obtained by the action of $\hat{\mathcal{L}}_-$:

$$(\hat{\mathcal{L}}_-)^{(j-m)} \Phi^j_{J,\kappa,j} = N_{J\kappa jm} \Phi^j_{J,\kappa,m}, \quad N_{J\kappa jm} = \text{constant}. \tag{5.32}$$

$\tilde{n}$ takes values $0, 1 \ldots n$. So $j = (\tilde{l} + \tilde{n})$ takes all integer value from κ to J:

$$j = \frac{\kappa}{2}, \frac{\kappa}{2} + 2, \frac{\kappa}{2} + 4 \ldots J. \tag{5.33}$$

Thus $\mathcal{H}_{nl}$ is spanned by the operators

$$\Phi^j_{J,\kappa,m} \qquad \text{with} \quad -j \leq m \leq j \quad j = \frac{\kappa}{2}, \frac{\kappa}{2} + 1, \ldots J. \tag{5.34}$$

An arbitary element Φ of $\mathcal{H}_{nl}$ can be expressed as

$$\Phi = \sum_{j=\frac{\kappa}{2}}^{J} \sum_{m=-j}^{j} c^j_{J,\kappa,m} \Phi^j_{J,\kappa,m}, \quad c^j_{J,\kappa,m} \in \mathbb{C}. \tag{5.35}$$

Any element Φ of $\mathcal{H}_{nl}$ is also an eigenfunction of the topological charge operator $\hat{K}_0$

$$\hat{K}_0 \equiv [\hat{N}, \quad], \qquad \hat{K}_0 \Phi \equiv [\hat{N}, \Phi] = \frac{\kappa}{2} \Phi. \tag{5.36}$$

Φ is thus the noncommutative analogue of a section of the complex line bundle with topological charge κ, which takes only even integer values ($\frac{\kappa}{2} \in \mathbb{Z}_+$).

A Appendix

Using the map (3.1), the condition (3.6) can be computed explicitly as

$$\begin{aligned}
y_i y_i &= -\big((\bar{z}_2^2 z_3^2 + \bar{z}_3^2 z_2^2 - 2\bar{z}_2\bar{z}_3 z_3 z_2) + (\bar{z}_3^2 z_1^2 + \bar{z}_1^2 z_3^2 - \\
&\quad 2\bar{z}_3\bar{z}_1 z_1 z_3) + (\bar{z}_1^2 z_2^2 + \bar{z}_2^2 z_1^2 - 2\bar{z}_1\bar{z}_2 z_2 z_1)\big) \\
&= -\big(\bar{z}_1^2(z_2^2 + z_3^2) + \bar{z}_2^2(z_1^2 + z_3^2) + \bar{z}_3^2(z_1^2 + z_2^2) + (\bar{z}_1^2 z_1^2 + \bar{z}_2^2 z_2^2 + \bar{z}_3^2 z_3^2) - \\
&\quad (\bar{z}_1^2 z_1^2 + \bar{z}_2^2 z_2^2 + \bar{z}_3^2 z_3^2) - (2\bar{z}_1\bar{z}_2 z_2 z_1 + 2\bar{z}_2\bar{z}_3 z_3 z_2 + 2\bar{z}_3\bar{z}_1 z_1 z_3)\big) \\
&= (\bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{z}_3 z_3)^2 - (\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2)(z_1^2 + z_2^2 + z_3^2) \\
&= (\bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{z}_3 z_3)^2 - \bar{\mathcal{O}}\mathcal{O} \qquad \text{(A.1)}
\end{aligned}$$

The fuzzy computation is also straightforward. Here we have to use the map (5.12). By direct substitution we get (5.14):

$$\begin{aligned}
\hat{C} = \hat{L}_i\hat{L}_i &= -((\hat{a}_2^\dagger\hat{a}_3\hat{a}_2^\dagger\hat{a}_3 + \hat{a}_3^\dagger\hat{a}_2\hat{a}_3^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_3\hat{a}_3^\dagger\hat{a}_2 - \hat{a}_3^\dagger\hat{a}_2\hat{a}_2^\dagger\hat{a}_3) + \\
&\quad (\hat{a}_3^\dagger\hat{a}_1\hat{a}_3^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_3\hat{a}_3^\dagger\hat{a}_1 - \hat{a}_3^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_3 + \hat{a}_1^\dagger\hat{a}_3\hat{a}_1^\dagger\hat{a}_3) + \\
&\quad (\hat{a}_1^\dagger\hat{a}_2\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2\hat{a}_2^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_1\hat{a}_2^\dagger\hat{a}_1)) \\
&= -(\hat{a}_1^\dagger\hat{a}_1^\dagger(\hat{a}_2\hat{a}_2 + \hat{a}_3\hat{a}_3) + \hat{a}_2^\dagger\hat{a}_2^\dagger(\hat{a}_3\hat{a}_3 + \hat{a}_1\hat{a}_1) + \hat{a}_3^\dagger\hat{a}_3^\dagger(\hat{a}_2\hat{a}_2 + \hat{a}_1\hat{a}_1)) + \\
&\quad \hat{a}_2^\dagger(\hat{a}_3^\dagger\hat{a}_3 + 1)\hat{a}_2 + \hat{a}_3^\dagger(\hat{a}_2^\dagger\hat{a}_2 + 1)\hat{a}_3 + \hat{a}_3^\dagger(\hat{a}_1^\dagger\hat{a}_1 + 1)\hat{a}_3 + \\
&\quad \hat{a}_1^\dagger(\hat{a}_3^\dagger\hat{a}_3 + 1)\hat{a}_1 + \hat{a}_2^\dagger(\hat{a}_1^\dagger\hat{a}_1 + 1)\hat{a}_2 + \hat{a}_1^\dagger(\hat{a}_2^\dagger\hat{a}_2 + 1)\hat{a}_1 \\
&= -(\hat{a}_1^\dagger\hat{a}_1^\dagger + \hat{a}_2^\dagger\hat{a}_2^\dagger + \hat{a}_3^\dagger\hat{a}_3^\dagger)(\hat{a}_1\hat{a}_1 + \hat{a}_2\hat{a}_2 + \hat{a}_3\hat{a}_3) + \\
&\quad \hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2^\dagger\hat{a}_2\hat{a}_2 + \hat{a}_3^\dagger\hat{a}_3^\dagger\hat{a}_3\hat{a}_3 + 2\hat{N}_2\hat{N}_3 + 2\hat{N}_3\hat{N}_1 + 2\hat{N}_1\hat{N}_2 + \\
&\quad 2(\hat{N}_1 + \hat{N}_2 + \hat{N}_3) \\
&= -\hat{\mathcal{O}}^\dagger\hat{\mathcal{O}} + \hat{a}_1^\dagger(\hat{a}_1\hat{a}_1^\dagger - 1)\hat{a}_1 + \hat{a}_2^\dagger(\hat{a}_2\hat{a}_2^\dagger - 1)\hat{a}_2 + \hat{a}_3^\dagger(\hat{a}_3\hat{a}_3^\dagger + 1)\hat{a}_3 + \\
&\quad 2\hat{N}_2\hat{N}_3 + 2\hat{N}_3\hat{N}_1 + 2\hat{N}_1\hat{N}_2 + 2(\hat{N}_1 + \hat{N}_2 + \hat{N}_3) + 2\hat{N} \\
&= -\hat{\mathcal{O}}^\dagger\hat{\mathcal{O}} + \hat{N}_1^2 - \hat{N}_1 + \hat{N}_2^2 - \hat{N}_2 + \hat{N}_3^2 - \hat{N}_3 \\
&\quad 2\hat{N}_2\hat{N}_3 + 2\hat{N}_3\hat{N}_1 + 2\hat{N}_1\hat{N}_2 + 2(\hat{N}_1 + \hat{N}_2 + \hat{N}_3) + 2\hat{N} \\
&= -\hat{\mathcal{O}}^\dagger\hat{\mathcal{O}} + (\hat{N}_1 + \hat{N}_2 + \hat{N}_3)^2 - \hat{N} + 2\hat{N} \\
&= \hat{N}(\hat{N} + 1) - \hat{\mathcal{O}}^\dagger\hat{\mathcal{O}}. \qquad \text{(A.2)}
\end{aligned}$$

The operators $\hat{A}_\alpha$ are defined as

$$\begin{pmatrix} \hat{A}_1 \\ \hat{A}_2 \\ \hat{A}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{pmatrix}. \qquad \text{(A.3)}$$

It can be easily shown that these operators $\hat{A}_\alpha$ satisfy the oscillator algebra:

$$[\hat{A}_1, \hat{A}_1^\dagger] = \frac{1}{2}([\hat{a}_1, \hat{a}_1^\dagger] + [\hat{a}_2, \hat{a}_2^\dagger]) = 1, \qquad \text{(A.4)}$$

$$[\hat{A}_1, \hat{A}_2] = 0, \qquad \text{(A.5)}$$

$$[\hat{A}_1, \hat{A}_2^\dagger] = \frac{1}{2}([\hat{a}_1, \hat{a}_1^\dagger] - [\hat{a}_2, \hat{a}_2^\dagger]) = 0. \qquad \text{(A.6)}$$

The total number operator can be reexpressed in terms of the operators $\hat{A}_\alpha$ as

$$\begin{aligned}\hat{N} &= \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_3^\dagger \hat{a}_3 \\ &= \left(\frac{\hat{A}_1^\dagger + \hat{A}_2^\dagger}{\sqrt{2}}\right)\left(\frac{\hat{A}_1 + \hat{A}_2}{\sqrt{2}}\right) + \left(i\frac{\hat{A}_1^\dagger - \hat{A}_2^\dagger}{\sqrt{2}}\right)\left(-i\frac{\hat{A}_1 - \hat{A}_2}{\sqrt{2}}\right) + \hat{A}_3^\dagger \hat{A}_3 \\ &= \hat{A}_1^\dagger \hat{A}_1 + \hat{A}_2^\dagger \hat{A}_2 + \hat{A}_3^\dagger \hat{A}_3. \end{aligned} \tag{A.7}$$

The operators $\hat{L}_i$ can also be rewritten in terms of the operators $\hat{A}_\alpha$ as

$$\begin{aligned}\hat{L}_+ &= \hat{L}_1 + i\hat{L}_2 \\ &= -i(\hat{a}_2^\dagger \hat{a}_3 - \hat{a}_3^\dagger \hat{a}_2) + i(-i(\hat{a}_3^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_3)) \\ &= \hat{a}_3^\dagger(\hat{a}_1 + i\hat{a}_2) - (\hat{a}_1^\dagger + i\hat{a}_2^\dagger)\hat{a}_3 \\ &= \sqrt{2}(\hat{A}_3^\dagger \hat{A}_1 - \hat{A}_2^\dagger \hat{A}_3), \end{aligned} \tag{A.8}$$

$$\begin{aligned}\hat{L}_- &= \hat{L}_1 - i\hat{L}_2 \\ &= -i(\hat{a}_2^\dagger \hat{a}_3 - \hat{a}_3^\dagger \hat{a}_2) - i(-i(\hat{a}_3^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_3)) \\ &= (\hat{a}_1^\dagger - i\hat{a}_2^\dagger)\hat{a}_3 - \hat{a}_3^\dagger(\hat{a}_1 - i\hat{a}_2) \\ &= \sqrt{2}(\hat{A}_1^\dagger \hat{A}_3 - \hat{A}_3^\dagger \hat{A}_2), \end{aligned} \tag{A.9}$$

$$\begin{aligned}\hat{L}_3 &= -i(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1) \\ &= -\frac{i}{2}((\hat{A}_1^\dagger + \hat{A}_2^\dagger)(-i)(\hat{A}_1 - \hat{A}_2) - i(\hat{A}_1^\dagger - \hat{A}_2^\dagger)(\hat{A}_1 + \hat{A}_2)) \\ &= -(\hat{A}_1^\dagger \hat{A}_1 - \hat{A}_2^\dagger \hat{A}_2). \end{aligned} \tag{A.10}$$

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